Student Number ____________________

GOSFORD HIGH SCHOOL

2020
YEAR 12

TRIAL HSC EXAMINATION

MATHEMATICS EXTENSION TWO

Duration- 3 hours plus 10 minutes reading time

START A NEW PAGE FOR EACH QUESTION
WRITE ON ONLY ONE SIDE OF THE PAGE

Multiple choice	**/10**
Question 11	**/15**
Question 12	**/15**
Question 13	**/15**
Question 14	**/15**
Question 15	**/15**
Question 16	**/15**
Total	**/100**

Section I

10 marks
Attempt questions 1 – 10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for questions 1 – 10

1 Let $m, n \in \mathbb{Z}$. Which of the following statements is false?

(A) n is even if and only if $n + 1$ is odd.

(B) $m + n$ is odd if and only if $m - n$ is odd.

(C) $m + n$ is even if and only if m and n are even.

(D) m and n are odd if and only if mn is odd.

2 The algebraic fraction $\frac{x}{3(x+c)^2}$, where c is a non-zero real number, can be written in partial fraction form, where A and B are real numbers, as

(A) $\dfrac{A}{x+c} + \dfrac{B}{x+c}$

(B) $\dfrac{A}{3x+c} + \dfrac{B}{x+c}$

(C) $\dfrac{A}{x+c} + \dfrac{B}{(x+c)^2}$

(D) $\dfrac{A}{3x+c} + \dfrac{B}{(x+c)^2}$

3 Let $\underset{\sim}{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. The angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is

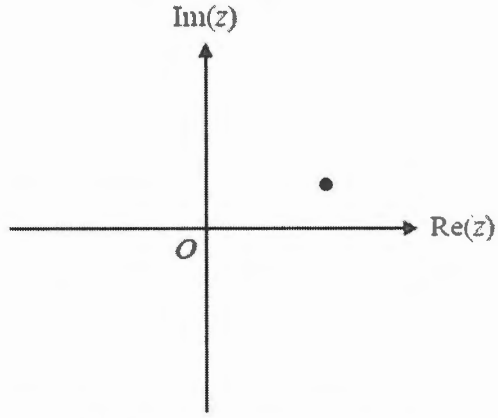
(A) 0°

(B) 45°

(C) 30°

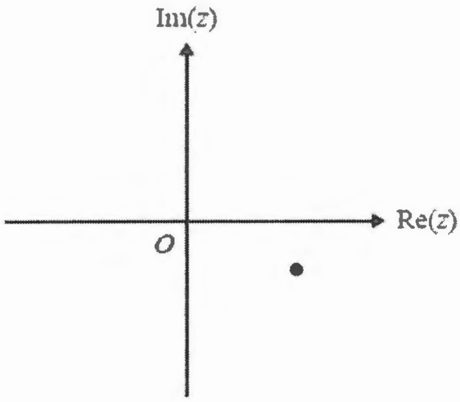
(D) 22.5°

4 The complex number $a + bi$, where a and b are real constants, is represented in the following diagram.

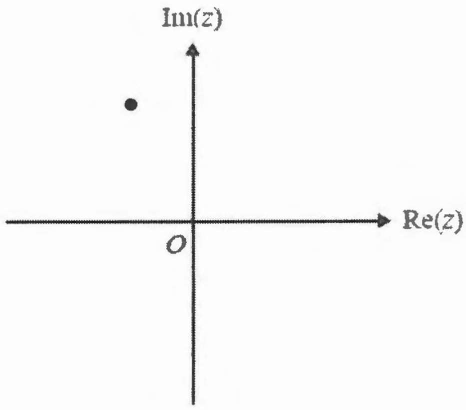


The complex number $-i(a + bi)$ could be represented by

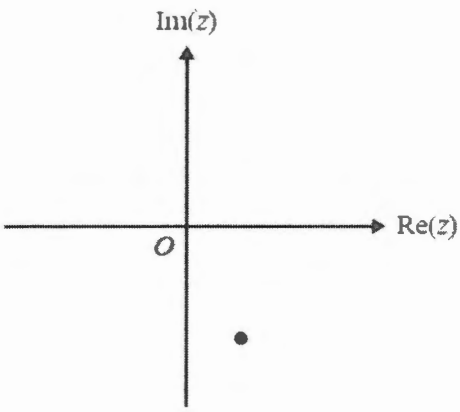
(A)



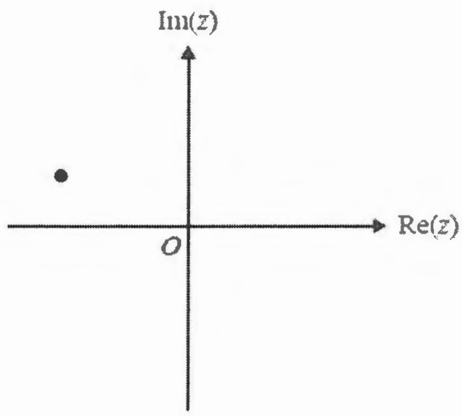
(B)



(C)



(D)



5 The equation, in Cartesian form, of the locus of the point z if $|z + 2i| = |z + 4|$ is:

(A) $x - 2y + 3 = 0$

(B) $2x - y + 3 = 0$

(C) $x + 2y + 3 = 0$

(D) $2x + y + 3 = 0$

6 Using a suitable substitution, $\int_a^b x(x^2 + 1)^5 \, dx$ is equal to

(A) $\frac{1}{2} \int_{a^2+1}^{b^2+1} u^5 \, du$

(B) $\frac{1}{2} \int_a^b u^5 \, du$

(C) $2 \int_{a^2+1}^{b^2+1} u^5 \, du$

(D) $2 \int_a^b u^5 \, du$

7 A unit vector perpendicular to $\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$ is

(A) $\frac{1}{4}\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$

(B) $\frac{1}{29}\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$

(C) $\frac{1}{\sqrt{30}}\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$

(D) $\frac{1}{\sqrt{29}}\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$

8 Which of the following is the complex number $z = 5\sqrt{3} - 5i$?

(A) $10e^{-\frac{i\pi}{6}}$

(B) $10e^{\frac{5\pi}{6}}$

(C) $5e^{-\frac{i\pi}{6}}$

(D) $5e^{\frac{5\pi}{6}}$

9 If a, b and c are any real numbers with $a > b$, which of the following statements must always be true?

(A) $\frac{1}{a} > \frac{1}{b}$

(B) $\frac{1}{a} < \frac{1}{b}$

(C) $ac > bc$

(D) $a + c > b + c$

10 A particle is describing SHM in a straight line with an amplitude of 3 metres. Its speed is 4 ms^{-1} when the particle is 1 metre from the centre of the motion.

What is the period of the motion?

(A) $\sqrt{3}\pi$

(B) $\sqrt{2}\pi$

(C) $\frac{\sqrt{3}\pi}{3}$

(D) $\frac{\sqrt{2}\pi}{2}$

END OF SECTION I

Section II

90 marks

Attempt questions 11 – 16

Allow about 2 hours and 45 minutes for this section

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) If $A = 6 - 4i$ and $B = -8 + 6i$, evaluate the following:

i) AB **1**

ii) $\frac{A}{B}$ **1**

iii) $\sqrt{B}$ **2**

(b) For a complex number z, $|z - 1| = 2|z + 1|$.

i) Find the cartesian equation of the locus of z. **3**

ii) Describe the locus of z. **2**

(c) The polynomial $P(x) = x^3 - 5x^2 + ax + b$, where a and b are real, has one root at $x = 3 - 2\sqrt{2}i$. Find the values of a and b. **3**

(d) For a complex number z, shade the region of the Argand Plane in which: $1 < |z| \leq 3$ and $-\frac{\pi}{3} \leq arg\, z \leq \frac{\pi}{4}$. **3**

Question 12 (15 marks) **Marks**

(a) The velocity $v\ ms^{-1}$ of a particle moving in simple harmonic motion along the x axis is given by $v^2 = 8 + 2x - x^2$.

i) Between which two points is the particle oscillating? **2**

ii) Find the acceleration of the particle in terms of x. **1**

iii) Find the period and amplitude of the motion. **2**

(b) i) Prove that $\frac{1}{10} - \frac{1}{11} < \frac{1}{100}$ **2**

ii) Let $n > 0$. Prove that $\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}$ **2**

(c) Consider this statement: If mn and $m + n$ are even, then m and n are even for $m, n \in \mathbb{Z}$.

i) Write down the contrapositive. **1**

ii) Prove the contrapositive. **3**

(d) Suppose that $c^2 - b^2 = 4$. Prove that b and c cannot both be positive integers. **2**

Question 13 (15 marks) **Marks**

(a) Find $\int \frac{dx}{2x\sqrt{\ln x}}$ **2**

(b) Find $\int \frac{dx}{x\sqrt{x^2-1}}$ using the substitution $u = \sqrt{x^2 - 1}$ **2**

(c) Find $\int_0^{\frac{\pi}{4}} x^2 \cos x \, dx$ by using integration by parts. **3**

(d) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{3+5\cos x}$ **3**

(e) (i) Find the real numbers A and B such that: **2**

$$\frac{3x^2 - x + 12}{(x-2)(x^2+2x+3)} \equiv \frac{A}{x-2} + \frac{Bx-3}{x^2+2x+3}$$

(ii) Hence find $\int \frac{3x^2-x+12}{(x-2)(x^2+2x+3)} \, dx$ **3**

Question 14 (15 marks) **Marks**

(a) Let $\underset{\sim}{u} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} a \\ 2 \\ -1 \end{pmatrix}$. Given that the vector projection of $\underset{\sim}{v}$ in the direction of $\underset{\sim}{u}$ is $\begin{pmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ -\frac{4}{9} \end{pmatrix}$, find the value of a. **2**

(b) Using vectors, show that the lines connecting any point on the semicircle $y = \sqrt{1 - x^2}$ to the points $(1, 0)$ and $(-1, 0)$ are perpendicular. **2**

(c) Three points P, Q and R have position vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $k(2\underset{\sim}{p} + \underset{\sim}{q})$ respectively, relative to a fixed origin O. The points O, P and Q are not collinear. Find the value of k if $\overrightarrow{QR}$ is parallel to $\underset{\sim}{p}$. **3**

(d) With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations:

$l_1 : \mathbf{r} = 9\mathbf{i} + 13\mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$ and $l_2 : \mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} + \mathbf{k})$

where λ and μ are scalar parameters.

i) Given that l_1 and l_2 meet, find the position vector of their point of intersection. **3**

ii) Find the acute angle between l_1 and l_2, correct to the nearest tenth of a degree. **2**

iii) Given that the point A has position vector $4\mathbf{i} + 16\mathbf{j} - 3\mathbf{k}$ and that the point $P(x_1, y_1, z_1)$ lies on l_1 such that AP is perpendicular to l_1, find the exact coordinates of P. **3**

Question 15 (15 marks) **Marks**

(a) i) Show that $(a^2 - b^2)(c^2 - d^2) \leq (ac - bd)^2$ **3**

ii) Deduce that $(a^2 - b^2)(a^4 - b^4) \leq (a^3 - b^3)^2$ **2**

(b) Find the point(s) of intersection of the line with parametric equation

$$r = i + 3j - 4k + t(i + 2j + 2k)$$

and the sphere with equation

$$(x - 1)^2 + (y - 3)^2 + (z + 4)^2 = 81.$$ **4**

(c) i) show that $k^2 + k$ is always even **3**

ii) Using the result in part (i), prove, by mathematical induction, that for all positive integral values of n, $n^3 + 5n$ is divisible by 6. **3**

Question 16 (15 marks)

(a) i) Show that for $I_n = \int x^n e^{-3x}\, dx$, $I_n = \frac{-x^n e^{-3x}}{3} + \frac{n}{3} I_{n-1}$ **3**

ii) Hence evaluate $\int x^3 e^{-3x}\, dx$. **3**

(b) i) Use De Moivre's Theorem to express $\cos 5\theta$ and $\sin 5\theta$ in terms of powers of $\sin \theta$ and $\cos \theta$. **2**

ii) Write an expression for $\tan 5\theta$ in terms of t, where $t = \tan\theta$. **1**

iii) By solving $\tan 5\theta = 0$, deduce that: $\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5$. **3**

(c) Prove that $33^n - 16^n - 28^n + 11^n$ is divisible by 85 for all positive integers $n \geq 2$. **3**

End of paper

Year 12 Extension 2 Trial 2020 – Solutions

Section I

1	2	3	4	5	6	7	8	9	10
C	C	B	C	B	A	D	A	D	B

Question 1

$m = 1, n = 3$ is a counterexample as m and n are both odd but $m + n = 1 + 3 = 4$ which is even.

Question 2

It just is!

Question 3

$Cos\ \theta = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}| \times |\underset{\sim}{v}|} = \frac{1\times1+1\times2+0\times2}{\sqrt{1^2+1^2+0^2}\times\sqrt{1^2+2^2+2^2}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$. Hence $\theta = \cos^{-1}\frac{1}{\sqrt{2}} = 45°$.

Question 4

Multiplication by $-i$ is equivalent to a rotation through $-\frac{\pi}{2}$ which is C.

A – is a reflection in the real axis, B – is a rotation through $\frac{\pi}{2}$, D – is a reflection in the imaginary axis.

Question 5

Let $z = x + iy$, then $|z + 2i| = |z + 4| \Rightarrow x^2 + (y + 2)^2 = (x + 4)^2 + y^2$

$\therefore x^2 + y^2 + 4y + 4 = x^2 + 8x + 16 + y^2 \Rightarrow 2x - y + 3 = 0$

Question 6

Let $u = x^2 + 1$. Then $du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}\,du$. At $x = a, u = a^2 + 1$ and $x = b, u = b^2 + 1$

Question 7

Let $\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} = 0$, hence $\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} \perp \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$.

D is a unit vector $\left(\text{if } \underset{\sim}{u} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} \text{ then } \frac{1}{|\underset{\sim}{u}|} = \frac{1}{\sqrt{2^2+(-4)^2+3^2}} = \frac{1}{\sqrt{29}}\right)$, B is not.

Question 8

Now $|z| = \sqrt{\left(5\sqrt{3}\right)^2 + (-5)^2} = 10$

Also $\arg(z) = \tan^{-1}\left(-\frac{5}{5\sqrt{3}}\right) = -\frac{\pi}{6}$

Question 9

Counterexamples – for A: if $a = 2$ and $b = 1$ then $2 > 1$ but $\frac{1}{2} < \frac{1}{1}$,

– for B: if $a = 2$ and $b = -1$ then $2 > -1$ but $\frac{1}{2} > -\frac{1}{1}$

– for C: if $a = 2, b = 1$ and $c = -1$ then $2 > 1$ but $2 \times -1 < 1 \times -1$

Question 10

If $v^2 = n^2(a^2 - x^2)$ then $16 = n^2(9-1) \Rightarrow n = \sqrt{2}$.

Hence the period is $\frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$

Section II

Question 11

(a) (i) $AB = (6-4i)(-8+6i) = -48 + 32i + 36i - 24i^2 = -24 + 68i$

(a) (ii) $\frac{A}{B} = \frac{6-4i}{-8+6i} \times \frac{-8-6i}{-8-6i} = \frac{-48+32i-36i+24i^2}{64+36} = \frac{-72-4i}{100}$

(a) (iii) Let $(x+iy)^2 = -8+6i$. Then $x^2 - y^2 + 2ixy = -8 + 6i$.

Hence $x^2 - y^2 = -8$ and $2xy = 6 \Rightarrow xy = 3 \Rightarrow y = \frac{3}{x}$

Hence $x^2 - \frac{9}{x^2} + 8 = 0 \Rightarrow x^4 + 8x^2 - 9 = 0 \Rightarrow x^2 = 1 \; or -9$.

Hence $x = 1, y = 3$ or $x = -1, y = -3$

Hence $\sqrt{B} = \pm(1+3i)$

(b) (i) Let $z = x + iy$.

Then $|z-1|^2 = 4|z+1|^2 \Rightarrow (x-1)^2 + y^2 = 4((x+1)^2 + y^2)$.

$\therefore x^2 - 2x + 1 + y^2 = 4x^2 + 8x + 4 + 4y^2$

Hence the locus of z is $3x^2 + 10x + 3y^2 = -3$

(b) (ii) $3x^2 + 10x + 3y^2 = -3 \Rightarrow x^2 + \frac{10}{3}x + y^2 = -1 \Rightarrow \left(x + \frac{5}{3}\right)^2 + y^2 = \frac{16}{9}$

Hence the locus is a circle with centre $\left(-\frac{5}{3}, 0\right)$ and radius $\frac{4}{3}$

(c) Another root of $P(x)$ is $3 + 2\sqrt{2}\,i$

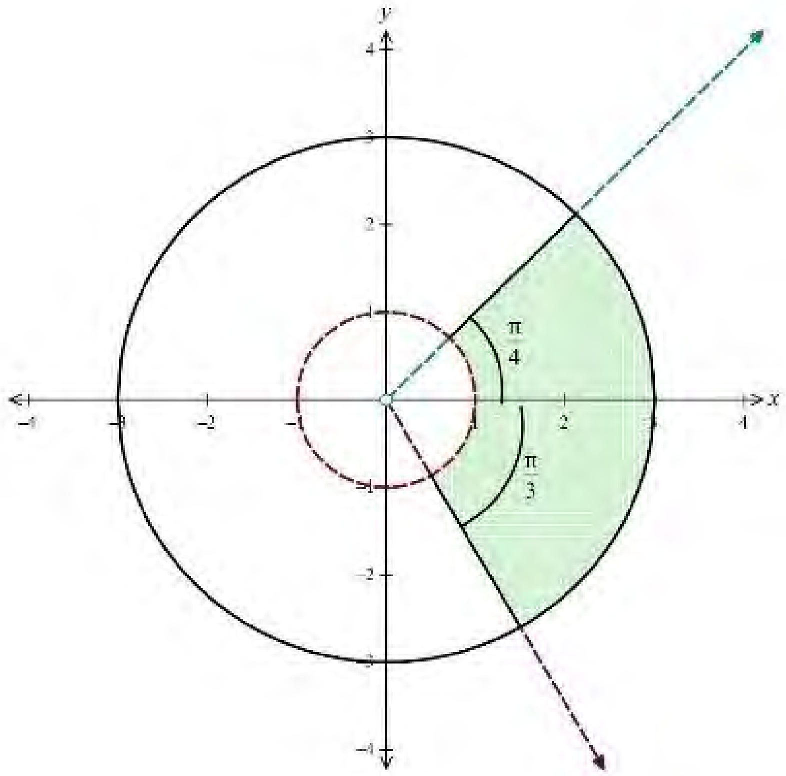
Now since the sum of the roots is 5, the third root must be -1.

Also, the product of the roots is $-b = (3 + 2\sqrt{2}\,i)(3 - 2\sqrt{2}\,i) \times -1 = -17$

And the product of the roots taken two at a time is $a = -(3 + 2\sqrt{2}\,i) - (3 - 2\sqrt{2}\,i) + 17 = 11$

Hence $a = 11, b = 17$.

(d)



Question 12

(a) i) Since $v^2 \geq 0$, then $8 + 2x - x^2 = (4 - x)(x + 2) \geq 0$.

Hence $-2 \leq x \leq 4$.

The particle oscillates between $x = -2$ and $x = 4$.

(a) ii) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(4 + x - \frac{1}{2}x^2\right) = 1 - x$

(a) iii) The distance between the endpoints is 6, hence the amplitude is 3.

$\ddot{x} = -1(x - 1)$, hence the period is $\frac{2\pi}{1} = 2\pi$.

(b) i) $\frac{1}{10} - \frac{1}{11} = \frac{110}{1100} - \frac{100}{1100} = \frac{10}{1100} < \frac{11}{1100} = \frac{1}{100}$

Hence $\frac{1}{10} - \frac{1}{11} < \frac{1}{100}$

(b) ii) $\frac{1}{n} - \frac{1}{n+1} = \frac{n(n+1)}{n^2(n+1)} - \frac{n^2}{n^2(n+1)} = \frac{n}{n^2(n+1)} < \frac{n+1}{n^2(n+1)} = \frac{1}{n^2}$

Hence $\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}$ for $n > 0$

(c) i) If m is odd or n is odd, then mn is odd or $m + n$ is odd.

(c) ii) Let m be odd and n be even i.e. $m = 2j + 1$ and $n = 2k, j, k \in \mathbb{Z}$.

Then $m + n = 2j + 1 + 2k$

$= 2(j + k) + 1$ which is odd.

Clearly this is also true if m is even and n is odd.

Let m be odd and n be odd i.e. $m = 2j + 1$ and $n = 2k + 1, j, k \in \mathbb{Z}$.

Then $mn = (2j + 1)(2k + 1)$

$= 4jk + 2j + 2k + 1$

$= 2(2jk + 2j + 2k) + 1$ which is odd.

Hence, if m is odd or n is odd, then mn is odd or $m + n$ is odd.

(d) Suppose that $c^2 - b^2 = 4$ and that b and c are both positive integers.

If $c^2 - b^2 = 4$ then $(c - b)(c + b) = 4$

As b and c are both positive integers, then $c - b$ and $c + b$ are factor pairs of 4.

Since $b \neq 0$, then $c - b = 1$ and $c + b = 4 \Rightarrow c = \frac{5}{2}$ which is not an integer.

Question 13

(a) Let $u = \ln x$. Then $du = \frac{1}{x} dx$

$$\int \frac{dx}{2x\sqrt{\ln x}} = \int \frac{u^{-\frac{1}{2}}}{2} du = u^{\frac{1}{2}} + c = \sqrt{\ln x} + c$$

(b) Let $u = \sqrt{x^2 - 1}$. Then $x = \sqrt{u^2 + 1}$ and $dx = \frac{u}{\sqrt{u^2+1}} du$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{1}{u\sqrt{u^2+1}} \times \frac{u}{\sqrt{u^2+1}} du = \int \frac{du}{u^2+1}$$

$$= \tan^{-1} u + c = \tan^{-1} \sqrt{x^2 - 1} + c$$

(c) Let $u = x^2,\ v' = \cos x$. Then $u' = 2x\ ,\ v = \sin x$.

$$\int_0^{\frac{\pi}{4}} x^2 \cos x \ dx = \Big[\ x^2 \sin x\ \Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 2x \sin x \ dx$$

$$= \frac{\pi^2}{16\sqrt{2}} - \left\{ \Big[\ -2x\cos x\ \Big]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} 2\cos x \ dx \right\}$$

$$= \frac{\pi^2}{16\sqrt{2}} + \frac{\pi}{2\sqrt{2}} - \Big[\ 2\sin x\ \Big]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi^2}{16\sqrt{2}} + \frac{\pi}{2\sqrt{2}} - \frac{2}{\sqrt{2}}$$

(d) Let $t = \tan\left(\frac{x}{2}\right)$. Then $\cos x = \frac{1-t^2}{1+t^2}$ and $dt = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)\ dx = \frac{1}{2}\left(\tan^2\left(\frac{x}{2}\right) + 1\right)\ dx \Rightarrow\ dx = \frac{2dt}{1+t^2}\ dt$

Also at $x = 0, t = 0$ and $x = \frac{\pi}{2}, t = 1$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{3+5\cos x} = \int_0^1 \frac{\frac{2dt}{1+t^2}}{3+5\left(\frac{1-t^2}{1+t^2}\right)} = \int_0^1 \frac{\frac{2dt}{1+t^2}}{\frac{3+3t^2+5-5t^2}{1+t^2}} = \int_0^1 \frac{2dt}{8-2t^2}$$

$$= \int_0^1 \frac{dt}{4-t^2} = \int_0^1 \left(\frac{1}{4(2-t)} + \frac{1}{4(2+t)}\right) dt$$

$$= \frac{1}{4}\Big[-\ln|2-t| + \ln|2+t|\Big]_0^1 = \frac{\ln 3}{4}$$

(e) (i) $\frac{3x^2-x+12}{(x-2)(x^2+2x+3)} \equiv \frac{A}{x-2} + \frac{Bx-3}{x^2+2x+3} \ \Rightarrow\ 3x^2 - x + 12 \equiv A(x^2+2x+3) + (Bx-3)(x-2)$

Let $x = 2$. Then $22 = 11A \ \Rightarrow\ A = 2$

Let $x = 1$. Then $14 = 2 \times 6 - (B-3) \ \Rightarrow\ B = 1$

(e) (ii) $\int \frac{3x^2-x+12}{(x-2)(x^2+2x+3)}\ dx = \int \frac{2}{x-2}\ dx + \int \frac{x-3}{x^2+2x+3}\ dx$

$$= 2\ln|x-2| + \int \frac{x+1-4}{x^2+2x+3}\ dx$$

$$= 2\ln|x-2| + \frac{1}{2}\ln|x^2+2x+3| - \int \frac{4}{(x+1)^2+2}\ dx$$

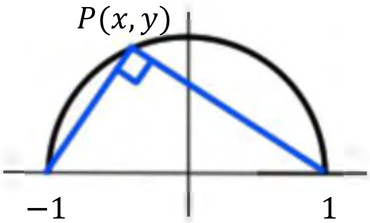
$$= 2\ln|x-2| + \frac{1}{2}\ln|x^2+2x+3| - 2\sqrt{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

Question 14

(a) Now the scalar projection of $\underset{\sim}{v}$ in the direction of $\underset{\sim}{u}$ is $\underset{\sim}{v} \cdot \underset{\sim}{\hat{u}} = \frac{2a-2+2}{\sqrt{2^2+(-1)^2+(-2)^2}} = \frac{2a}{\sqrt{9}} = \frac{2a}{3}$

Hence the vector projection of $\underset{\sim}{v}$ in the direction of $\underset{\sim}{u}$ is $\frac{2a}{3}\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ -\frac{4}{9} \end{pmatrix} \Rightarrow a = 1$

(b) Let $P(x, y)$ be a point on the semicircle $y = \sqrt{1 - x^2}$



Then the line segments are given by the vectors $\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} x+1 \\ y \end{pmatrix}$

and $\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x-1 \\ y \end{pmatrix}$.

Now $\begin{pmatrix} x+1 \\ y \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y \end{pmatrix} = (x+1)(x-1) + y^2$

$= x^2 + y^2 - 1 = x^2 + 1 - x^2 - 1 = 0$

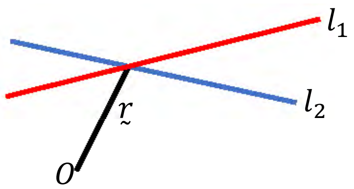
Hence the line segments are perpendicular.

(c) $\overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR} = -\underset{\sim}{q} + k(2\underset{\sim}{p} + \underset{\sim}{q}) = 2k\underset{\sim}{p} + (k-1)\underset{\sim}{q}$

If $\overrightarrow{QR}$ is parallel to $\underset{\sim}{p}$, then there is some $\lambda \in \mathbb{R} \setminus \{0\}$ such that $2k\underset{\sim}{p} + (k-1)\underset{\sim}{q} = \lambda\underset{\sim}{p}$

$\Rightarrow 2k = \lambda$ and $k - 1 = 0$. Hence $k = 1$

(d) i) At the point of intersection $\begin{pmatrix} 9 \\ 13 \\ -3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$



$\therefore 9 + \lambda = 2 + 2\mu \Rightarrow \lambda = 2\mu - 7$ (†)

and $13 + 4\lambda = -1 + \mu \Rightarrow \mu = 4\lambda + 14$ (‡)

Sub (†) into (‡): $\mu = 4(2\mu - 7) + 14 = 8\mu - 14$

$\therefore 7\mu = 14 \Rightarrow \mu = 2 \Rightarrow \lambda = 2 \times 2 - 7 = -3$

$\therefore \mathbf{r} = 9\mathbf{i} + 13\mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = 6\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

(d) ii) Let $\underset{\sim}{a} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}| \times |\underset{\sim}{b}|} = \frac{1\times2+4\times1+(-2)\times1}{\sqrt{1^2+4^2+(-2)^2} \times \sqrt{2^2+1^2+1^2}} = \frac{4}{\sqrt{21} \times \sqrt{6}}$

$\therefore \theta = \cos^{-1}\left(\frac{4}{\sqrt{126}}\right) = 69.1°$

(d) iii) Let $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 9 \\ 13 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} \Rightarrow x_1 = 9 + \lambda,\ y_1 = 13 + 4\lambda$ and $z_1 = -3 - 2\lambda$,

Now $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \begin{pmatrix} x_1 - 4 \\ y_1 - 16 \\ z_1 + 3 \end{pmatrix} = \begin{pmatrix} 5 + \lambda \\ 4\lambda - 3 \\ -2\lambda \end{pmatrix}$

But $(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \cdot \overrightarrow{AP} = 0 \Rightarrow 1(5 + \lambda) + 4(4\lambda - 3) - 2 \times -2\lambda = 0 \Rightarrow \lambda = \frac{1}{3}$

Hence P is $\left(\frac{28}{3}, \frac{43}{3}, -\frac{11}{3}\right)$.

Question 15

(a) i) $(a^2 - b^2)(c^2 - d^2) - (ac - bd)^2 = a^2c^2 - b^2c^2 - a^2d^2 + b^2d^2 - a^2c^2 + 2abcd - b^2d^2$

$= -b^2c^2 - a^2d^2 + 2abcd = -(ad - bc)^2 \leq 0$

Hence $(a^2 - b^2)(c^2 - d^2) \leq (ac - bd)^2$

(a) ii) $(a^2 - b^2)(a^4 - b^4) - (a^3 - b^3)^2 = a^6 - b^2a^4 - a^2b^4 + b^6 - a^6 + 2a^3b^3 - b^6$

$= -b^2a^4 - a^2b^4 + 2a^3b^3 = -(ab^2 - a^2b)^2 \leq 0$

Hence $(a^2 - b^2)(a^4 - b^4) \leq (a^3 - b^3)^2$

(b) $r = i + 3j - 4k + t(i + 2j + 2k)$

$x = 1 + t$

$y = 3 + 2t$

$z = -4 + 2t$

Now $(x - 1)^2 + (y - 3)^2 + (z + 4)^2 = 81$

$(1 + t - 1)^2 + (3 + 2t - 3)^2 + (-4 + 2t + 4)^2 = 81$

$(t)^2 + (2t)^2 + (2t)^2 = 81$

$9t^2 = 81$

$t^2 = 9 \Rightarrow \quad t = \pm 3$

$\therefore$ Points are: $[1 + 3, 3 + 2(3), -4 + 2(3)] = (4, 9, 2)$

$[1 - 3, 3 + 2(-3), -4 + 2(-3)] = (-2, -3, -10)$

(c) i) If $k$is even, i.e $k = 2x$ for some $x \in \mathbb{Z}$, then

$k^2 + k \quad = (2x)^2 + 2x = 4x^2 + 2x$

$= 2(2x^2 + x) = 2y$ for some $y \in \mathbb{Z}$

If $k$is odd, i.e $k = 2x + 1$ for some $x \in \mathbb{Z}$, then

$k^2 + k \quad = (2x + 1)^2 + 2x + 1 = 4x^2 + 4x + 1 + 2x$

$= 4x^2 + 6x + 2 = 2(2x^2 + 3x + 1) = 2y$ for some $y \in \mathbb{Z}$

Hence $k^2 + k$ is even.

(c) ii) $n = 1$: $1^3 + 5 \times 1 = 6$ which is divisible by 6.

Assume that $n^3 + 5n$ is divisible by 6 for $n = k$

i.e. $k^3 + 5k = 6p$ where p is an integer.

$n = k + 1$: $(k + 1)^3 + 5(k + 1) = k^3 + 3k^2 + 3k + 1 + 5k + 5 = k^3 + 5k + 3k^2 + 3k + 6$

$= 6p + 3k^2 + 3k + 6 = 6p + 6 + 3(k^2 + k)$

$= 6p + 6 + 3(2y) = 6(p + y + 1)$ [from part (i)]

Hence $(k + 1)^3 + 5(k + 1)$ is divisible by 6

$\therefore$ if true for $n = k$, then also true for $n = k + 1$, but since true for $n = 1$, by induction is true for all integral values, $n \geq 1$.

Question 16

(a) i) $I_n = \int x^n e^{-3x} \, dx$

$u = x^n \qquad v' = e^{-3x}$

$u' = nx^{n-1} \qquad v = -\frac{1}{3} e^{-3x}$

$I_n = uv - \int vu'$

$I_n = (x^n)\left(-\frac{1}{3} e^{-3x}\right) - \int \left(-\frac{1}{3} e^{-3x}\right)(nx^{n-1}) \, dx$

$I_n = \frac{-x^n e^{-3x}}{3} + \int \frac{nx^{n-1}}{3} e^{-3x} \, dx$

$I_n = \frac{-x^n e^{-3x}}{3} + \frac{n}{3} \int x^{n-1} e^{-3x} \, dx$

$\therefore I_n = \frac{-x^n e^{-3x}}{3} + \frac{n}{3} I_{n-1}$

(a) ii) $\int x^3\, e^{-3x}\, dx \quad = \frac{-x^3 e^{-3x}}{3} + \frac{3}{3}\int x^2\, e^{-3x}\, dx$

$\int x^2\, e^{-3x}\, dx \quad = \frac{-x^2 e^{-3x}}{3} + \frac{2}{3}\int x^1\, e^{-3x}\, dx$

$\frac{2}{3}\int x^1\, e^{-3x}\, dx \quad = \frac{2}{3}\left[\frac{-x^1 e^{-3x}}{3}\right] + \frac{2}{3}\times\frac{1}{3}\int x^0\, e^{-3x}\, dx$

$= \frac{-2xe^{-3x}}{9} + \frac{2}{9} \times -\frac{1}{3}e^{-3x}$

$= \frac{-2xe^{-3x}}{9} - \frac{2}{27}\, e^{-3x}$

$\therefore \int x^3\, e^{-3x}\, dx$

$= \frac{-x^3 e^{-3x}}{3} - \frac{x^2 e^{-3x}}{3} - \frac{2xe^{-3x}}{9} - \frac{2}{27}\, e^{-3x} + C$

(b) i) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

LHS $= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

Equating Real and Imaginary Parts: $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

(b) ii) $\tan 5\theta = \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta} = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta} = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$

(b) iii) If $\tan 5\theta = 0$, then $5\theta = 0, \pi, 2\pi, 3\pi, 4\pi \Rightarrow \theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$

Also if $\tan 5\theta = 0$, then $\frac{5t-10t^3+t^5}{1-10t^2+5t^4} \Rightarrow 5t - 10t^3 + t^5 = t(t^4 - 10t^2 + 5) = 0$

Hence the roots of $t^4 - 10t^2 + 5$ are $t = \tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, \tan\frac{3\pi}{5}, \tan\frac{4\pi}{5}$

Product of Roots of $t^4 - 10t^2 + 5$ are $\tan\frac{\pi}{5} \times \tan\frac{2\pi}{5} \times \tan\frac{3\pi}{5} \times \tan\frac{4\pi}{5} = 5$

(c) $33^n - 16^n - 28^n + 11^n = (33^n - 16^n) - (28^n - 11^n)$

$= (33 - 16)(33^{n-1} + 33^{n-2} \times 16 + \cdots + 16^{n-1}) - (28 - 11)(28^{n-1} + 28^{n-2} \times 11 + \cdots + 11^{n-1})$

$= 17(33^{n-1} + 33^{n-2} \times 16 + \cdots + 16^{n-1}) - 17(28^{n-1} + 28^{n-2} \times 11 + \cdots + 11^{n-1})$

Hence $33^n - 16^n - 28^n + 11^n$ is divisible by 17.

Also, $33^n - 16^n - 28^n + 11^n = (33^n - 28^n) - (16^n - 11^n)$

$= (33 - 28)(33^{n-1} + 33^{n-2} \times 28 + \cdots + 28^{n-1}) - (16 - 11)(16^{n-1} + 16^{n-2} \times 11 + \cdots + 11^{n-1})$

$= 5(33^{n-1} + 33^{n-2} \times 28 + \cdots + 28^{n-1}) - 5(16^{n-1} + 16^{n-2} \times 11 + \cdots + 11^{n-1})$

Hence $33^n - 16^n - 28^n + 11^n$ is divisible by 5.

Thus $33^n - 16^n - 28^n + 11^n$ is divisible by 17 and by 5.

Therfore $33^n - 16^n - 28^n + 11^n$ is divisible by $17 \times 5 = 85$ as 17 and 5 are prime numbers.